9 Scientific models and mathematical equations

> **Key words**: equation, algebraic equation, formula, expression, variable, constant, coefficient, brackets, order of operations, subject of a formula, proportional, directly proportional, constant of proportionality, linear relationship, linear equation, inversely proportional, exponential relationship, inverse square relationship, line graph, rate, intercept, gradient, tangent, area under the line (on a graph).

In science, a graph shows a relationship between quantities in the real world. Some graphs are produced by collecting and plotting experimental data. However, some graphs are representations of how we might imagine the world to be, based on certain sets of assumptions or 'scientific models'. Underlying the behaviour of these models are the mathematical equations that breathe life into our imagined worlds.

9.1 Equations, formulae and expressions

Before going further, it is useful to clarify the meaning of a number of terms about which there is sometimes confusion. Central to this chapter will be a discussion of the manipulation and graphical representation of **_equations_** (also referred to as **_algebraic equations_**). An equation is a mathematical statement that indicates the equality of the expressions to the left and right of the equals ($=$) sign. Figure 9.1 shows some examples of equations.

All of these equations contain variables but they differ in their nature. In equations (a), (b) and (c) the variables are _abstract_, but in equations (d), (e) and (f) the variables represent _physical quantities_. An equation that shows the relationship between physical quantities is called a **_formula_**. So, every formula is an equation, though not every equation is a formula.

Note that although equation (d) looks different to equation (a) because it uses _words_ (density, mass and volume) rather than _symbols_ (x and y), this is _not_ what makes it a formula. Equations (e) and (f) both represent the same formula for calculating kinetic energy. What makes both of them formulae is that the variables, whether expressed as words or as symbols, relate to physical quantities. (The advantages and disadvantages of using words or symbols are discussed in [Section 9.5](#) _The real-world meaning of a formula_ on page 93.)

> **Figure 9.1** Examples of equations
>
> (a) $\quad y = \dfrac{x}{3} - 1$
>
> (b) $\quad y = mx + c$
>
> (c) $\quad 2x^2 + 5x - 3 = 0$
>
> (d) $\quad \text{density} = \dfrac{\text{mass}}{\text{volume}}$
>
> (e) $\quad \text{kinetic energy} = \frac{1}{2} \times \text{mass} \times \text{velocity}^2$
>
> (f) $\quad E_k = \frac{1}{2}mv^2$

Pupils should also be aware that the terms *formula* and *equation* have additional meanings in chemistry: a chemical *formula* (e.g. H_2O) is a symbolic way of showing the relative numbers of atoms in a substance, while a chemical *equation* is a symbolic representation of the rearrangement of those atoms that occurs in a chemical reaction. The idea of 'balance' applies to both types of equation: in an algebraic equation the *values* on each side are equal, and in a chemical equation the *numbers of atoms* on each side are equal.

An **_expression_** is a combination of numbers and variables that may be evaluated – expressions do *not* contain the equals (=) sign. (Note that 'evaluating an expression' means finding its numerical value – a very different meaning to 'evaluating a science investigation'.) So, the equations listed above are not themselves expressions, but they do *contain* expressions. Examples of expressions are shown in Figure 9.2.

Figure 9.2 Examples of expressions

$$\frac{x}{3} - 1 \qquad \frac{x}{3} \qquad 3x^2 - 5x - 2 \qquad 5x \qquad \frac{mass}{volume} \qquad velocity^2 \qquad \tfrac{1}{2}mv^2$$

Notice that, in an algebraic expression such as $5x$, the convention is that there is no multiplication sign between the '5' and the '*x*', even though this means '5 multiplied by *x*'. However, for a word expression, the multiplication sign is included for clarity, for example $\tfrac{1}{2} \times mass \times velocity^2$.

When such an expression is expressed symbolically, the multiplication signs are omitted and it is written as $\tfrac{1}{2}mv^2$ (writing $\tfrac{1}{2} \times m \times v^2$ could be confusing as $\times$ could be mistaken for *x*). Since these symbols do not have a space between them, all physical quantities are represented by just a single letter (e.g. *m* for mass, *v* for velocity, and so on). Additional information about a variable that needs to be included can be indicated using a subscript or superscript (e.g. E_k). Note that this contrasts with units, which often have more than one letter (e.g. cm or kg) and are always written with spaces in between each unit.

Similarly, using the division sign ($\div$) explicitly is not the only way to represent division. The following expressions are the same:

$$mass \div volume \qquad mass \,/\, volume \qquad \frac{mass}{volume}$$

The third way of representing division in an expression is generally preferable because it makes the relationship clearer to see. This is particularly the case when there are more than two values or variables in the expression.

9.2 Variables, constants and coefficients

Expressions may include letters that represent *variables*, *constants* and *coefficients*, and it is also useful to clarify the meaning of these words, particularly as they are used differently in mathematics and science.

An example of a **_variable_** is represented by the letter *x* in this **_equation_**:

$$y = \frac{x}{3} - 1$$

It is a variable because it can take on a range of values. In this example, as the value of x varies, y also varies, and so y is a variable too.

In science, the term 'variable' is also used in this sense, being represented by a letter or word(s) in an algebraic equation. For example, in the following equation, *mass* and *volume* are both variables and could each take on a range of values:

$$\text{density} = \frac{\text{mass}}{\text{volume}}$$

These values would give a range of values for *density*, and so this too is a variable. However, the term 'variable' is used in a broader sense in science. It can be used to refer to *any* factor that could be varied in a scientific investigation, whether or not it forms part of an algebraic equation. Such equations represent only *__quantitative__* variables, and often variables are *__qualitative__* (categorical). Quantitative variables (continuous or discrete) may be identified at the start of an investigation, to find out whether there is a relationship that can be expressed as an algebraic equation; even if none is found, they are still referred to as 'variables'.

Note that in published texts the letters that represent variables are shown in *italics*, but not the letters that represent units. So, mass may be represented by the letter m, while the abbreviation for metre is m. This distinction is not made when writing by hand.

An example of the use of a *__constant__* in mathematics is illustrated in the equation:

$$y = mx + c$$

This is the general equation of a straight line, where m and c represent constants (m is the gradient of the line and c is the intercept). Substituting different numerical values for m and c gives different straight lines; for example, $y = 2x + 1$ represents one particular straight line, and $y = 3x + 2$ represents another one.

In a scientific investigation, we may refer to '*keeping a variable constant*' (i.e. the control variable). For example, the current through an electrical resistor depends on two variables – its resistance and the voltage applied across it. In an investigation, we could look at the effect on the current of changing the resistance while keeping the voltage constant, or changing the voltage while keeping the resistance constant.

The word 'constant' is also used in science to refer to those physical quantities that really are 'constant', and where they always have the same value whenever they are used. Examples of such physical constants include the speed of light in a vacuum (about $3 \times 10^{8}\,\text{m s}^{-1}$) and the Avogadro constant (about $6.02 \times 10^{23}\,\text{mol}^{-1}$), Note that although these are *constants*, this does not mean they are just numbers – they are values that have *units*.

The word *__coefficient__* can easily be confused with *constant*. In the expression $3x^2$, the coefficient of x^2 is '3', and in the expression $5x$, the coefficient of x is '5'. The term does not just apply to numerical values; so, for example, in the expression $\frac{1}{2}mv^2$, as well as saying that ½ is the coefficient of mv^2, we could say that ½m is the coefficient of v^2. In science, however, the word is also often applied to a value that is constant for a particular material under certain conditions but that is different for different materials (e.g. the coefficient of expansion or the coefficient of thermal conductivity).

9.3 Operations and symbols

An **_expression_** may contain symbols that represent *operations*. Two common operations are *addition* and *subtraction*, and these are represented by the familiar plus ($+$) and minus ($-$) signs. *Multiplication* and *division* are also common operations, though the signs that can be used to represent them ($\times$ and $\div$) are often not used explicitly (see Section 9.1 *Equations, formulae and expressions* on page 87).

A subtle point, but one that becomes much more important for post-16 science, is that the plus ($+$) and minus ($-$) signs are actually used in two distinct ways. For example, take these two expressions:

$$5 - 3 \qquad -3$$

In the first of these, the minus symbol is acting as an *operator* – it is telling us to *subtract* the value 3 from the value 5. In the second of these, it is telling us that this is a *negative value*. The same applies to the 'plus' sign, which has different meanings in the following two expressions.

$$5 \quad 3 \qquad +3$$

Note that while the 'minus' symbol is *always* used to indicate a negative value, often we do not explicitly use the 'plus' symbol to indicate a positive value, and simply write '3'.

This distinction is essential in understanding expressions that involve the addition and subtraction of positive and negative values, for example:

$$(+6) + (-4) - (-2)$$

One area of 11–16 science where pupils may encounter such expressions is the use of vectors to describe and analyse motion (see Sections 10.5–10.8).

In an **_equation_**, the equals ($=$) sign indicates that the expressions on each side are equal. There are a number of other useful symbols that are used to compare expressions, and these are shown in Figure 9.3.

The first three symbols ($=$, $>$, $<$) are clear-cut in their meaning and are probably the ones most commonly encountered.

The next two symbols ($\geq$, $\leq$) can be useful in defining class intervals (see Section 6.4 *Displaying larger sets of values* on page 53). For example, the phrase 'those pupils whose height is 150 cm or above but less than 160 cm' can be expressed more simply as $150 \text{ cm} \leq \text{height} \leq 160 \text{ cm}$.

An approximate value can be indicated by using the symbol '~'; for example, approximately 3 g can be written as ~3 g. The symbol '$\approx$' is a combination of '$=$' and '~' so, instead of writing 'mass $= $~3 g', it is simpler to write 'mass $\approx$ 3 g'.

Figure 9.3 Examples of symbols used to compare expressions

$y = x$	'y equals x' or 'y is equal to x'
$y > x$	'y is greater than x'
$y < x$	'y is less than x'
$y \geq x$	'y is greater than or equal to x'
$y \leq x$	'y is less than or equal to x'
$y \approx x$	'y is approximately equal to x'
$y \gg x$	'y is much greater than x'
$y \ll x$	'y is much less than x'

The last two symbols ($\gg$, $\ll$) are not common in 11–16 science. One example of their use might be in a situation where one is handling an algebraic equation that includes an expression such as $m_A + m_B$, where these represent the masses of two objects, A and B. If the mass of A is very much bigger than the mass of B then the expression might be simplified, by

assuming that the mass of B can be ignored and that the total mass can be taken as just the mass of A. This can be represented as:

$$\text{since } m_A \gg m_B, \text{ then } m_A + m_B \approx m_A$$

9.4 Calculations using formulae: order of operations

Many of the *__formulae__* that pupils encounter in 11–16 science involve just one operation. For example, density can be calculated by dividing mass by volume – a single operation. In formulae where there is *more* than one operation, it is essential that they are carried out in the correct order.

Here is a simple example to illustrate this:

$$4 + 2 \times 3$$

A different value is obtained depending on whether the addition or multiplication is done first, as shown in Figure 9.4.

It might seem common sense that the first of these is correct since the operations are done in order from left to right, and the result should be 18. Indeed, if this series of numbers and symbols were entered into most calculators, the result given would be in fact be 18. However, this is *not* the convention that has been adopted in mathematics, in which *multiplication takes precedence over addition*. The correct value is therefore 10.

Figure 9.4 Which order is correct?

(a) Addition first	**(b)** Multiplication first
$4 + 2 \times 3$	$4 + 2 \times 3$
6×3	$4 + 6$
18	10

Pupils need to be aware of how to handle the *__order of operations__* in order to be able to make calculations and to rearrange formulae. The explanations for these will be given and then summarised at the end of this section.

In an *__expression__* involving only addition and subtraction, the operations are carried out in order from left to right (Figure 9.5a). A different order, for example, from right to left, may give a different (and incorrect) result (Figure 9.5b).

Figure 9.5 Addition and subtraction only

(a) Left to right (correct)	**(b)** Right to left (incorrect)
$4 + 3 - 2 - 1$	$4 + 3 - 2 - 1$
$7 - 2 - 1$	$4 + 3 - 1$
$5 - 1$	$4 + 2$
4	6

In an expression involving only multiplication and division (using $\times$ and $\div$ signs), these operations are also carried out in order from left to right (Figure 9.6a). As before, a different order may give an incorrect result (Figure 9.6b). However, this expression could be written more clearly and less ambiguously by avoiding the use of the $\div$ sign (Figure 9.6c). The top and bottom expressions are evaluated first before the final division.

Figure 9.6 Multiplication and division only

(a) Left to right (correct)	**(b)** Right to left (incorrect)	**(c)** Clearer (and correct)
$3 \times 4 \div 2 \div 2$	$3 \times 4 \div 2 \div 2$	$\dfrac{3 \times 4}{2 \times 2}$
$12 \div 2 \div 2$	$3 \times 4 \div 1$	
$6 \div 2$	3×4	$\dfrac{12}{4}$
3	12	3

If an expression contains a combination of these operations then multiplication and division take precedence over addition and subtraction (Figure 9.7a). If, however, a calculation requires that an addition or subtraction should take precedence then this can be done using *brackets*. (Figure 9.7b). Evaluating expressions in brackets takes precedence over *all* other operations.

> **Figure 9.7** Handling brackets
>
> **(a)** With no brackets
>
> $10 - 2 \times 3$
> $10 - 6$
> 4
>
> **(b)** With brackets
>
> $(10 - 2) \times 3$
> 8×3
> 24

It would be possible to add brackets to $10 - 2 \times 3$ giving the expression $10 - (2 \times 3)$, and it would still give a value of 4. The brackets, though, are unnecessary since multiplication already has precedence. Even though they are not needed, it would not be incorrect to use the brackets here, so it may be better to include brackets when in doubt or for additional clarity.

Pupils should also be able to handle calculations involving *indices* (i.e. those that include expressions of the form x^n in which x is raised to the power of n). Indices take precedence over all of the operations discussed so far (addition, subtraction, multiplication, division) except brackets. Figure 9.8a illustrates the precedence of an index over a multiplication, while Figure 9.8b shows how brackets take precedence over an index.

> **Figure 9.8** Handling indices
>
> **(a)** Index takes precedence
>
> 2×3^2
> 2×9
> 18
>
> **(b)** Brackets take precedence
>
> $(2 \times 3)^2$
> 6^2
> 36

To summarise these ideas, a rather contrived example of a calculation is shown in Figure 9.9 to illustrate the *order of operations*.

> **Figure 9.9** Order of operations
>
> | | |
> |---|---|
> | The original expression: | $4 + 2 \times (5 - 2)^2$ |
> | Expressions inside *brackets* are evaluated first, to give: | $4 + 2 \times 3^2$ |
> | Next *indices* are evaluated, to give: | $4 + 2 \times 9$ |
> | Then *multiplication* and *division*, to give: | $4 + 18$ |
> | Finally, *addition* and *subtraction*, to give: | 22 |

This convention is summarised in the widely used mnemonic BIDMAS (**B**rackets, **I**ndices, **D**ivision/**M**ultiplication, **A**ddition/**S**ubtraction). An alternative form of the mnemonic is BODMAS (where O represents **O**rder or 'to the power **O**f').

For simplicity, all of these examples have involved only numbers but the same conventions about the order of operations apply to the manipulation of algebraic equations. Furthermore, in scientific formulae the values have units; making sure that the handling of the units makes sense provides an additional check on the correct sequence of operations (e.g. one cannot add unlike units).

The use of calculators needs care in making sure that the operations are done in the correct order. It is often safer to write down intermediate values – indeed, in scientific calculations, these intermediate values are often useful to calculate in any case as they have a real-world meaning.

9.5   The real-world meaning of a formula

It is helpful to think of a ***formula*** not just as a mathematical equation but as something that 'tells a story' about the real world. For example, take the formula that defines speed:

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

As well being able to substitute values and to calculate a result, pupils should be able to interpret what this formula is saying and to check that this makes sense. The formula shows that speed is ***directly proportional*** to distance: so, in real-world terms, the greater the distance that someone walks in a certain time, the greater their speed. It also shows that speed is ***inversely proportional*** to time: so, the greater the time that someone walks a certain distance, the lower their speed. Pupils should be able to see that these real-world interpretations of the formula make sense.

The formula above defines the relationship between *three* variables. Knowing the values of any *two* of the variables means that the *third* variable can be calculated. The formula allows *speed* to be calculated but rearranging it gives formulae that allow *distance* and *time* to be calculated (how to rearrange formulae will be discussed in subsequent sections of this chapter):

$$\text{distance} = \text{speed} \times \text{time}$$

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

Again, in these formulae, pupils should be able to identify directly proportional and inversely proportional relationships, and to relate the formulae to real-world interpretations.

There are many formulae used in 11–16 science that involve three variables that are related in this way, i.e. through direct and inverse proportion. Other examples include:

$$\text{density} = \frac{\text{mass}}{\text{volume}}$$

$$\text{chemical amount (in moles)} = \frac{\text{mass of substance}}{\text{molar mass}}$$

$$\text{force exerted on spring} = \text{spring constant} \times \text{extension}$$

Note that the underlying form of all these formulae is the same. Although the last formula looks different from the others (the right-hand side shows two variables multiplied together), the first two formulae could also each be rearranged to show two variables multiplied together. Each of the formulae here, however, is shown in the way it is most commonly written.

Note also that some formulae represent *definitions* whereas others represent *empirical relationships*. For example, the first formula is a *definition*: it represents the way that density is defined. In a definition, the relationship between the variables is *exact*. The last formula, by contrast, represents an *empirical relationship* (Hooke's Law), which is an approximation to the way that real springs behave.

When pupils first start to use scientific formulae, it is generally better to express these using *words* rather than symbols for the variables, since this helps to emphasise the real-world

meaning. As they get older, it is appropriate that they also become familiar with symbolic formulae. These have a number of advantages. They are shorter to write down, making them easier to manipulate and rearrange. The symbolic form may also be easier to remember: in the two versions for kinetic energy below, the symbolic one is visually more recognisable, as well as its sound ('a-half-em-vee-squared') being more memorable.

$$\text{kinetic energy} = \tfrac{1}{2} \times \text{mass} \times \text{speed}^2 \quad E_k = \tfrac{1}{2}mv^2$$

The following sections discuss the techniques that can be used to rearrange formulae. Pupils need to be able to rearrange a formula if the quantity that they are trying to calculate is not 'on its own' on the left-hand side. The discussion starts with the simplest kind of formula, involving only addition and subtraction, before moving on to those involving multiplication and division.

9.6 Rearranging formulae involving addition and subtraction

Suppose that in a class of pupils there are 13 boys and 15 girls. It is not difficult to work out that the total number of pupils in the class is 28. More formally, one could represent this as a '*formula*' for calculating the number of pupils in a class (Figure 9.10).

Figure 9.10 A simple formula

number of pupils = number of boys + number of girls

$$= 13 + 15$$
$$= 28$$

Now suppose that, in a different class, we want to work out the number of girls knowing that the total number of pupils is 30 and the number of boys is 14. Again, it is not difficult to work out the result – there must be 16 girls – but the above formula does not give this directly. In order to do this, the formula needs to be *rearranged* so that 'number of girls' becomes the **subject of the formula**, i.e. it is 'on its own' and by convention on the *left* of the equals sign. Using our common sense about the situation, we should be able to write down a rearranged formula for working out the number of girls (Figure 9.11).

Figure 9.11 Rearranging the formula

number of girls = number of pupils – number of boys

$$= 30 - 14$$
$$= 16$$

However, most formulae are not as easy to rearrange as this, so it is important for pupils to understand the general principles for rearranging formulae, in order to apply these to any situation. There are really just two principles – for simplicity, these will be illustrated using only numbers at first but they apply in exactly the same way to formulae involving variables. Figure 9.12 uses the example of '2 + 3 = 5': the value of the expression on the left is 5 and that on the right is 5 – they are equal.

The first principle is that the sides of an equation can be *swapped* – an equation shows that the expression on the left is equal to the expression on the right, so it does not matter in

which order they are written. The left and right sides are still equal if the sides are swapped (Figure 9.12a).

The second principle is that the left and right sides of an equation remain equal if the *same operation is performed on each side*. For example, the sides are still equal to each other if the *same value is added* to each side: Figure 9.12b shows that, if '2' is added to each side, they remain equal (each has a value of 7). The same applies if the *same value is subtracted* from each side: if 3 is subtracted from each side, they are both equal to 2.

> **Figure 9.12** Principles for rearranging equations
>
> **(a)** Swapping sides
> 2 + 3 = 5
> the sides remain equal if the sides are swapped:
> 5 = 2 + 3
>
> **(b)** Doing the same thing to each side
> 2 + 3 = 5
> the sides remain equal if the same operation is performed on each side:
> 2 + 3 + 2 = 5 + 2
> 2 + 3 − 3 = 5 − 3

Returning to the original formula for working out the total number of pupils in a class, how could we rearrange this so that 'number of boys' is the subject of the formula (i.e. on the left)? Figure 9.13 shows how the two principles for rearranging equations can be applied to do this.

> **Figure 9.13** Applying the principles
>
> The original formula:
> number of pupils = number of boys + number of girls
> So that 'number of boys' is on the left, swap sides:
> number of boys + number of girls = number of pupils
> In order to have just 'number of boys' on the left side, subtract 'number of girls' from each side:
> number of boys + number of girls − number of girls = number of pupils − number of girls
> This gives:
> number of boys = number of pupils − number of girls

Writing out the steps like this might seem a bit laborious (though if symbols were used for the variables rather than words, it would be both quicker to write as well as clearer to see). However, manipulating equations in this way emphasises the *understanding of the principles*, which is important for equations where it may not be so straightforward. After gaining in confidence and understanding, some pupils might begin to take shortcuts, but it is not recommended that these should be taught as this can lead to misconceptions. It is better to teach in a way that focuses on the principles in order to develop understanding.

After rearranging a formula, it is always important to check it and to think about whether it *makes sense* (see Section 9.5 *The real-world meaning of a formula* on page 93).

9.7 Rearranging formulae involving multiplication and division

Many *formulae* in 11–16 science involve three variables that are *directly proportional* or *inversely proportional* to each other. For example, the formula that defines density is:

$$\text{density} = \frac{\text{mass}}{\text{volume}}$$

Suppose that you know the density and the volume of something and want to use the formula to calculate its mass. It needs to be rearranged so that *mass* is the *subject of the formula*. Rearranging such formulae is something that pupils find quite challenging.

Working first with just numbers may help pupils to explore and get a better sense of the different ways of expressing the relationship, for example:

$\frac{12}{6} = 2$ could be rearranged as $2 \times 6 = 12$ or $6 = \frac{12}{2}$ and so on.

Rearranging the formula for density uses the same two principles as in the previous section on addition and subtraction. Figure 9.14 shows how the principles of swapping sides and carrying out the same operation on both sides can be used to make *mass* the subject of the formula.

Figure 9.14 Rearranging to make mass the subject of the formula

The original formula:

$$\text{density} = \frac{\text{mass}}{\text{volume}}$$

So that 'mass' is on the left, swap sides:

$$\frac{\text{mass}}{\text{volume}} = \text{density}$$

To remove volume from the left side, multiply each side by 'volume':

$$\text{volume} \times \frac{\text{mass}}{\text{volume}} = \text{volume} \times \text{density}$$

On the left side, 'volume' cancels out (since volume ÷ volume = 1), and so the rearranged formula becomes:

$$\text{mass} = \text{volume} \times \text{density}$$

Suppose instead that we want to rearrange the original formula so that *volume* is the subject of the formula. This is shown in Figure 9.15. As always, pupils should check the meaning of a rearranged formula: does it make sense that volume is directly proportional to mass and inversely proportional to density? Would the formula be obviously wrong if these were reversed (so that density was divided by mass)?

Again, confident pupils might take shortcuts, but it is recommended that teaching should always emphasise an understanding of the principles by carrying out all of the steps.

9.8 Rearranging other formulae

Most *formulae* in 11–16 science involve only addition, subtraction, multiplication and division. One exception is the formula for kinetic energy. Suppose we want to rearrange this to make 'speed (*v*)' the *subject of the formula*.

$$E_{k} = \tfrac{1}{2}mv^2$$

Figure 9.15 Rearranging to make volume the subject of the formula

The original formula:

$$\text{density} = \frac{\text{mass}}{\text{volume}}$$

Here, swapping sides is not a helpful first step to get 'volume' on its own on the left side. Instead, multiply each side by 'volume':

$$\text{volume} \times \text{density} = \text{volume} \times \frac{\text{mass}}{\text{volume}}$$

On the right side, 'volume' cancels out (since volume ÷ volume = 1):

$$\text{volume} \times \text{density} = \text{mass}$$

In order to remove 'density' from the left side, divide each side by 'density':

$$\frac{\text{volume} \times \text{density}}{\text{density}} = \frac{\text{mass}}{\text{density}}$$

On the left side, 'density' cancels out (since density ÷ density = 1), and so the rearranged formula becomes:

$$\text{volume} = \frac{\text{mass}}{\text{density}}$$

This is a bit more difficult than the previous examples, but again illustrates the same two principles for rearranging equations (Figure 9.16).

Figure 9.16 Rearranging to make velocity the subject of the formula

The original formula:

$$E_k = \tfrac{1}{2}mv^2$$

So that v is on the left, swap sides:

$$\tfrac{1}{2}mv^2 = E_k$$

To remove the $\tfrac{1}{2}$, multiply both sides by 2:

$$mv^2 = 2E_k$$

Divide both sides by m:

$$v^2 = \frac{2E_k}{m}$$

Taking square roots of each side gives the final formula (this is another example of 'doing the same thing to both sides'):

$$v = \sqrt{\frac{2E_k}{m}}$$

Another example, this time involving reciprocals, is the formula for the total resistance of two resistors in parallel:

$$\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2}$$

There are a number of different ways that this could be rearranged so that R_{total} is the subject, though they all give the same result. Figure 9.17 shows one way of doing this.

Figure 9.17 Rearranging to make total resistance the subject of the formula

Original formula:

$$\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2}$$

Multiply both sides by $R_1 R_2 R_{total}$

$$\frac{R_1 R_2 R_{total}}{R_{total}} = R_1 R_2 R_{total}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

This simplifies to:

$$R_1 R_2 = R_2 R_{total} + R_1 R_{total}$$

Rearrange the expression on the right-hand side:

$$R_1 R_2 = R_{total}(R_1 + R_2)$$

Swap sides:

$$R_{total}(R_1 + R_2) = R_1 R_2$$

Divide both sides by $(R_1 + R_2)$:

$$R_{total} = \frac{R_1 R_2}{R_1 + R_2}$$

Although this example is considerably more demanding than the previous examples, it still uses the same two principles for rearranging equations.

9.9 Calculations without formulae

The idea that a *__formula__* tells a 'story' also works the other way round. It is useful to remember formulae, but knowing things about the way the world works means that formulae can often be worked out. In addition, it may not always be necessary to use a formula. This section gives two examples to illustrate this.

Example 1

What is the mass of $20\,cm^3$ of aluminium (density $= 2.7\,g/cm^3$)? One way of answering this question would be to write down the relevant formula (from memory or looking it up) and then substitute the values. Alternatively, it can be done by reasoning about the situation:

(1) $1\,cm^3$ of aluminium has a mass of $2.7\,g$

(2) $20\,cm^3$ of aluminium has a mass of $20 \times 2.7\,g = 54\,g$

Step (1) is using the original information to explain what the density implies. In step (2), the reasoning is that 20 times the volume of aluminium will have 20 times the mass.

Example 2 (which requires more steps):

What chemical amount (in moles) of water molecules are there in $10\,g$ of water (molar mass $= 18\,g/mol$)?

(1) $18\,g$ of water contains $1\,mol$ of water molecules

(2) $1\,g$ of water contains $1/18\,mol$ of water molecules

(3) $10\,g$ of water contains $10/18\,mol$ of water molecules $= 0.56\,mol$

Step (1) is using the original information to explain what the molar mass implies. In step (2), the reasoning is that 1/18 of the mass of water will have 1/18 of the chemical amount (in moles). Finally, in step (3), similar reasoning means that 10 times the mass of water has 10 times the chemical amount.

This technique is an example of *__proportional__* reasoning. Since it involves a step in which you calculate the value of one variable when the other has a numerical value of 1, it is known as the *unitary method*.

9.10 Use of 'calculation triangles'

A quite common technique for avoiding the need to rearrange formulae is the use of 'calculation triangles'. Many teachers dislike this method as they see it as a way of getting the right answer in an examination without any need for real understanding. Pupils often like the method for precisely the same reason, and some teachers may therefore feel under an obligation to use the method. The technique is generally regarded as poor practice because it does not encourage pupils to develop their understanding of these kinds of relationship.

An example of its use can be illustrated with the following question: What is the current through a resistance of $10\,\Omega$ if a potential difference of $3\,V$ is applied across it?

If this calculation is done using equations then the first step is to write down the formula that relates the three variables:

$$\text{potential difference} = \text{current} \times \text{resistance}$$

The next step is to rearrange this so that current is the subject of the formula:

$$\text{current} = \frac{\text{potential difference}}{\text{resistance}}$$

Substituting the values for potential difference (3 V) and resistance (10 Ω) gives the current as 0.3 A.

How to do this with a calculation triangle is shown in Figure 9.18. The relationship between potential difference (V), current (I) and resistance (R) is represented in Figure 9.18a. Covering up any one of the symbols in this triangle gives the expression required to calculate it. For example, to calculate the value of the current, the symbol 'I' is covered up (Figure 9.18b). Substituting the values in the expression for the remaining symbols (Figure 9.18c) gives the required answer.

Figure 9.18 Using 'calculation triangles' does not encourage understanding

(a) A calculation triangle   **(b)** Covering up the symbol 'I' …   **(c)** … gives the expression for calculating it
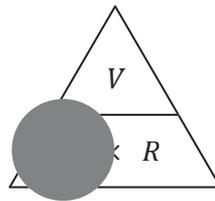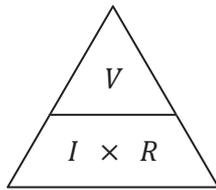
Of course, in order to use a 'calculation triangle', a pupil first needs to write it down with the three symbols in the correct positions. One of the problems with this method is that this is not the way that formulae are shown in scientific texts, nor the way the pupils are expected to remember them. So, remembering the correct calculation triangle requires at least as much work as remembering the formula.

Even once the triangle is written down, the use of this representation focuses more on just getting the right answer. As discussed in Section 9.5, pupils should always be thinking about the real-world meaning of a formula.

There are a variety of formulae in school science but calculation triangles have limited applicability and pupils may not always appreciate this. If they try to use triangles for equations that involve addition and subtraction, they will get incorrect results. Relying on their use means that pupils are not developing the skills to become fluent in rearranging different types of equations.

On a positive note, the visual form of a calculation triangle does emphasise that the three variables are related to each other, and that any of the variables can be calculated from the other two. A formula shows just one of these calculations. If teachers do feel pressurised into using them, they should be used as a complement to the understanding of the nature and meaning of equations rather than as a replacement.

9.11 Mathematical equations and relationships in science

Many relationships in science can be modelled by a small number of mathematical *equations*. Figures 9.19–9.24 show the most common of these, and how they can be represented on *line graphs*. Each figure shows the relevant mathematical equation (expressed using x and y), along with an example of where such a relationship can be found in science.

Figure 9.19 shows a *proportional* relationship (or a *directly proportional* relationship). This is a particularly common relationship in science and is discussed in detail in Chapter 5 *Working with proportionality and ratio*. The graph shows a straight line that passes through the *origin*.

An example of this is a resistor that follows Ohm's Law, in which the current through it is proportional to the potential difference applied across it. This means that, for example, if the potential difference is doubled then the current also doubles.

Figure 9.19 Proportional relationship: $y = mx$

$y = 2x$
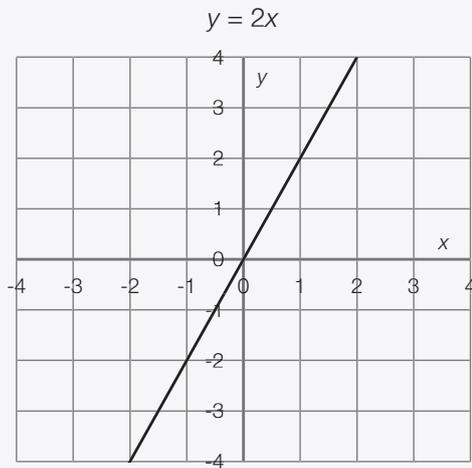
For a resistor that follows Ohm's Law, the current through it is *proportional* to the potential difference applied across it
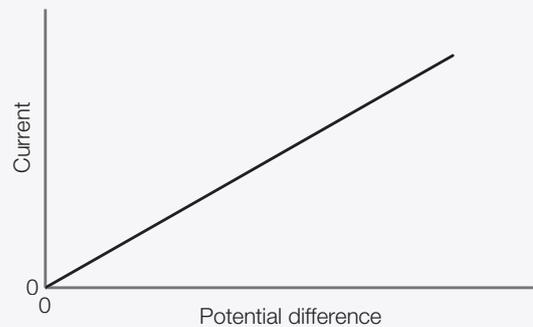
Figure 9.20 shows a ***linear relationship***. This is similar to a proportional relationship in that the graph shows a straight line, but here it does not pass through the origin. An example of this is Hooke's Law, in which the total length of a spring increases linearly with the force exerted on it. This means that equal increases in force produce equal increases in the length of the spring. The ***intercept*** on the vertical axis is the length of the spring when the force on it is zero, i.e. the 'normal' length of the spring.

Note that the general equation for a proportional relationship is often written as $y = kx$, where k is the ***constant of proportionality***. In Figure 9.19 it is written as $y = mx$, in order to emphasise the similarity to the general equation for a linear relationship, $y = mx + c$, as shown in Figure 9.20. A proportional relationship is a special case of a linear relationship in which $c = 0$. Since c is the intercept on the vertical axis, this means that, when it is zero, the line passes through the origin.

Figure 9.20 Linear relationship: $y = mx + c$

$y = \dfrac{1}{2}x + 1$
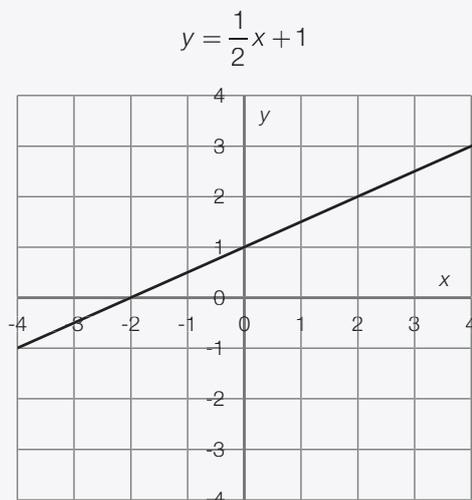
For a spring that follows Hooke's Law, the total length of the spring increases *linearly* with the force exerted on the spring
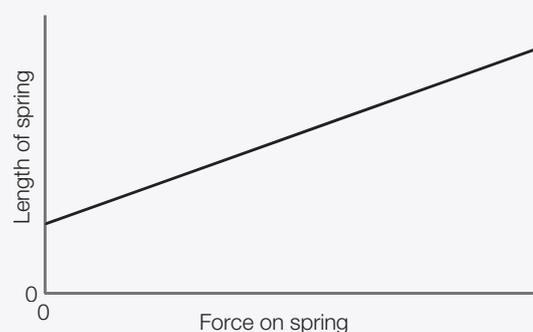
In Figure 9.20, the *length* of the spring is plotted on the vertical axis; subtracting its 'normal' length from these values gives the *extension* of the spring. Plotting these values would 'shift the line down' so that it passes through the origin. Instead of a linear relationship, this would then represent a proportional relationship. Extension is proportional to force and when there is no force the extension is zero.

Figure 9.21 shows a *square* relationship. Note here that the graph on the left includes both positive and negative values of *x*, while the science example just shows the right side of the graph representing only positive values. The example here is the relationship between the kinetic energy of an object and its speed (for which negative values would have no real-world meaning). This relationship is not linear: the line on the graph is curved, and it shows that the kinetic energy increases more rapidly than the speed.

Figure 9.21 Square relationship: $y = ax^2$

$y = x^2$

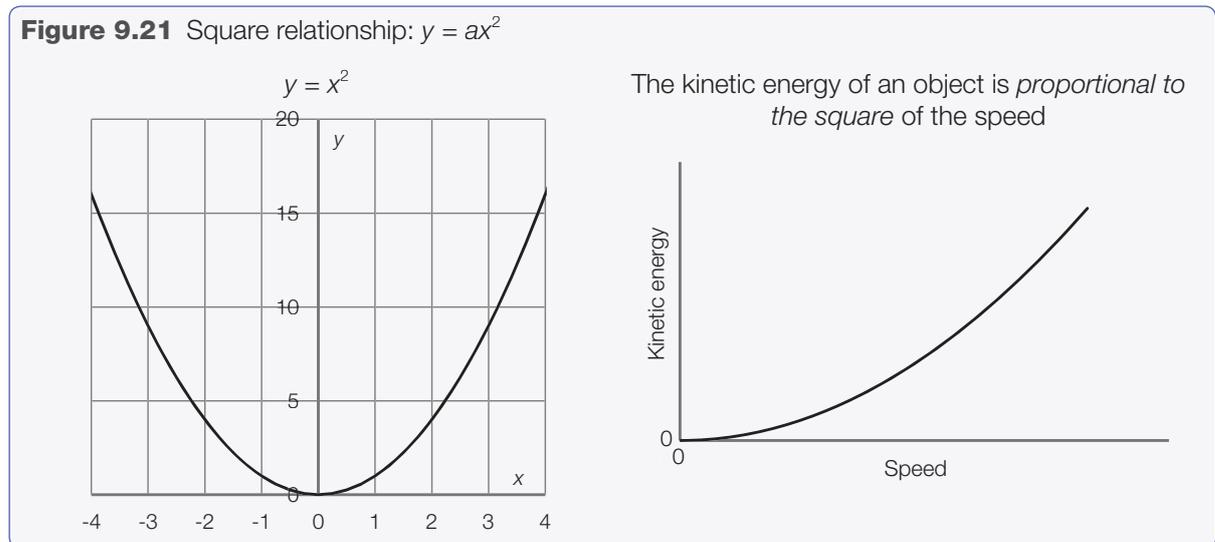The kinetic energy of an object is *proportional to the square* of the speed

Figure 9.22 shows an ***inversely proportional*** relationship (or an *inverse* relationship). This kind of relationship is also discussed in Chapter 5 *Working with proportionality and ratio*. Again, note that the graph on the left shows both positive and negative values of *x*. An example in science is the relationship between the volume and pressure of a fixed mass of gas. The inverse relationship means that, for example, if the pressure is *doubled* then the volume is *halved*. Note that, as the pressure is increased, the volume gets smaller and smaller but never reaches zero (it would if the pressure were infinite but this is impossible). On the graph, therefore, the curve gets closer and closer to the horizontal axis but never actually meets it. (The technical term for the line to which a curve is tending is an *asymptote*.)

Figure 9.23 shows an ***exponential relationship***. The rising curve on the graph looks similar to the curve for the square relationship but, in fact, an exponential curve rises much more rapidly than a square relationship does. Exponential relationships are found whenever the rate of change of a quantity is proportional to the quantity itself. For example, if the numbers of bacteria double every hour then, starting with 1 bacterium, there would be just 2 at the end of the first hour. In the fifth hour, there would be 16 at the start which would rise to 32. This leads to very rapid growth – if they continue to increase like this then there would be over 16 million at the end of 24 hours. In reality, there would be limits to the growth of increasing numbers of bacteria so, unlike the graph on the left, the curve cannot go on rising forever.

Figure 9.22 Inversely proportional relationship: $y = \dfrac{a}{x}$

$$y = \frac{2}{x}$$
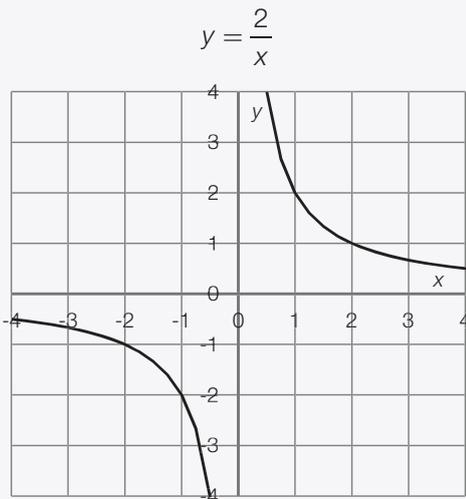
The volume of a fixed mass of gas is *inversely proportional* to the pressure (at constant temperature)
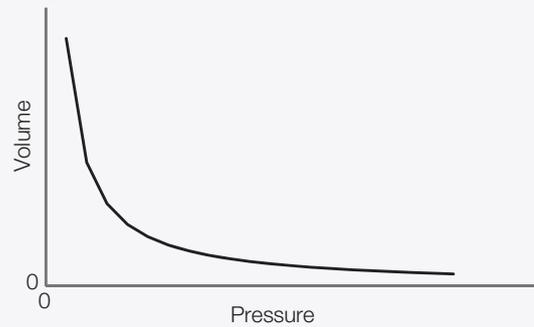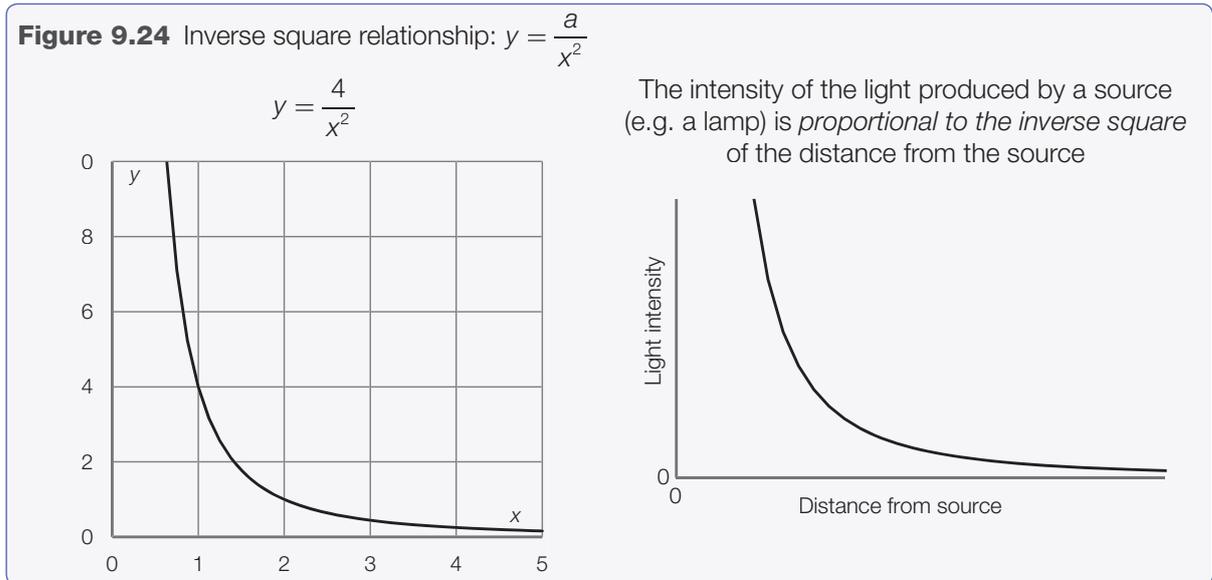
Figure 9.23 Exponential relationship: $y = a^x$

$$y = 2^x$$
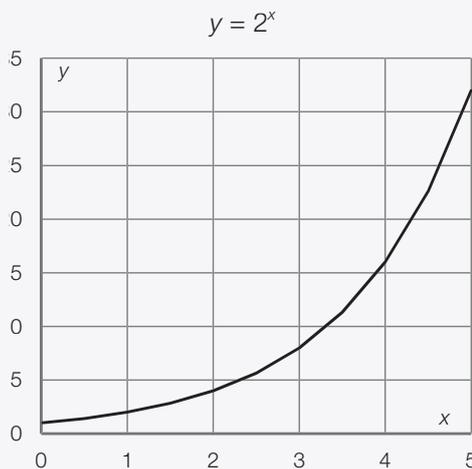
If the doubling time for bacteria is constant then the number of bacteria increases *exponentially*
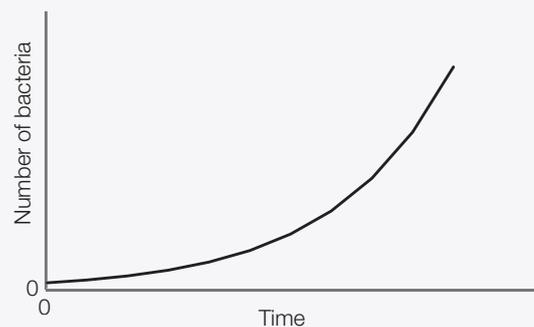
In the case of bacterial growth, the exponent x in the equation $y = a^x$ is greater than 1 and the change is an example of *exponential growth*. In radioactive decay, the rate of decay is proportional to the amount of radioactive material remaining but in this case the exponent is *smaller* than 1. The graph slopes downwards, rapidly at first and then slowly approaching the horizontal axis. This is an example of *exponential decay*.

Figure 9.24 shows an ***inverse square relationship***. This is similar in shape to the inverse relationship but the decrease towards the horizontal axis is rather steeper in this case. An example is the way that the intensity (or irradiance) of light from a lamp decreases as you move away from the lamp. Again, the curve approaches the horizontal axis but never meets it. So, as you move away from a lamp, the light intensity falls quite steeply but theoretically would never drop to zero, no matter how far you moved away.

Figure 9.24 Inverse square relationship: $y = \dfrac{a}{x^2}$

$$y = \frac{4}{x^2}$$

The intensity of the light produced by a source (e.g. a lamp) is *proportional to the inverse square* of the distance from the source

9.12 Graphs of quantities against time: gradients

A *__line graph__* shows the relationship between two variables. The way the line rises or falls tells us about how fast or slow the change is – i.e. about the *__rate__* of change of one variable with another. This section looks at line graphs that show changes over *time* (i.e. time is the variable on the horizontal axis); such line graphs tend to be the easiest to interpret because the way we talk about the horizontal axis on a graph often reflects a sense of 'one thing happening after another' in going from left to right. However, the principles discussed here apply to any kind of a line graph.

Figure 9.25 shows two graphs that represent a bath filling with water. In Figure 9.25a, the bath starts with 50 litres of water (the *__intercept__* on the vertical axis) and reaches 200 litres after 10 minutes. The straight line shows that the bath is filling up at a *constant rate*. In Figure 9.25b, the bath also starts with 50 litres and reaches 200 litres after 10 minutes, but here it is not filling up steadily – the *rate changes*. At the beginning it fills up more quickly, and then slows down towards the end.

Figure 9.25 Graphs of quantities against time: a bath filling with water

(a) | **(b)**
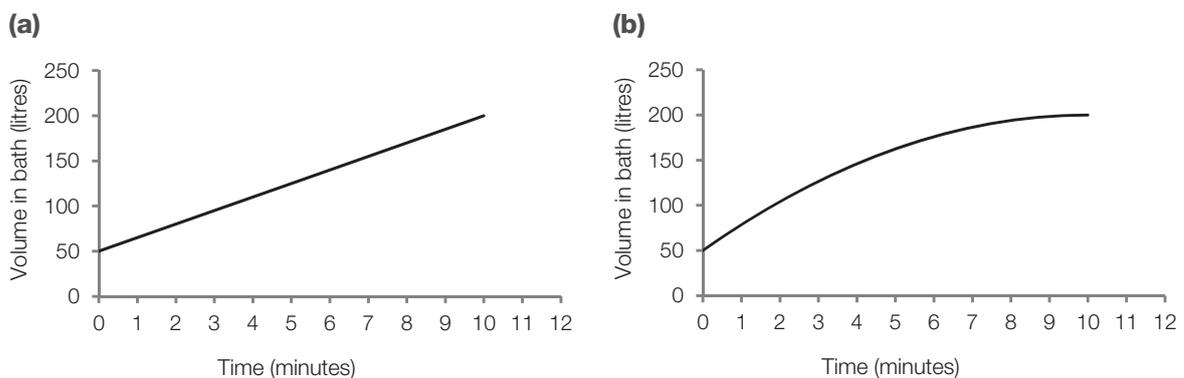
The *__gradient__* of the line represents the rate of change. Pupils need to be able to calculate the gradient of a line on a graph plotted by hand on graph paper. Figure 9.26a shows how this is done for a straight line graph. Finding the gradient involves finding the value for the change in the quantity on the horizontal axis and the corresponding change in the quantity on the vertical axis, and then dividing one by the other. The changes in the quantities can be found

by drawing a triangle, as shown in Figure 9.26a. It is always best to draw the triangle as large as possible (so that the values can be measured more accurately), while at the same time choosing a convenient value along the horizontal (in this case 10 minutes).

If pupils have drawn a line of best fit, they need to understand why drawing a triangle on the fitted line to calculate a gradient is better than just using the two extreme data points. Each of the data points is subject to measurement uncertainty, so the fitted line is the 'best guess' of the nature of the relationship.

The change along the vertical axis is 150 litres (200 litres − 50 litres). The gradient is then found by dividing the vertical value by the horizontal value:

$$\text{gradient} = \frac{150 \text{ litres}}{10 \text{ minutes}} = 15 \text{ litres/minute}$$

The gradient represents the rate at which water is flowing into the bath. In this example, the flow rate is constant: 15 litres are added to the bath every minute.

In the other example, shown in Figure 9.26b, the gradient of the curve changes over time. For example, the rate of change is greater at 3 minutes than at 7 minutes. This can be emphasised by drawing a **_tangent_** to the curve at each of these points. The gradient of the tangent at 3 minutes is steeper than the one at 7 minutes.

Figure 9.26 Finding the value of a rate by calculating the gradient of a line
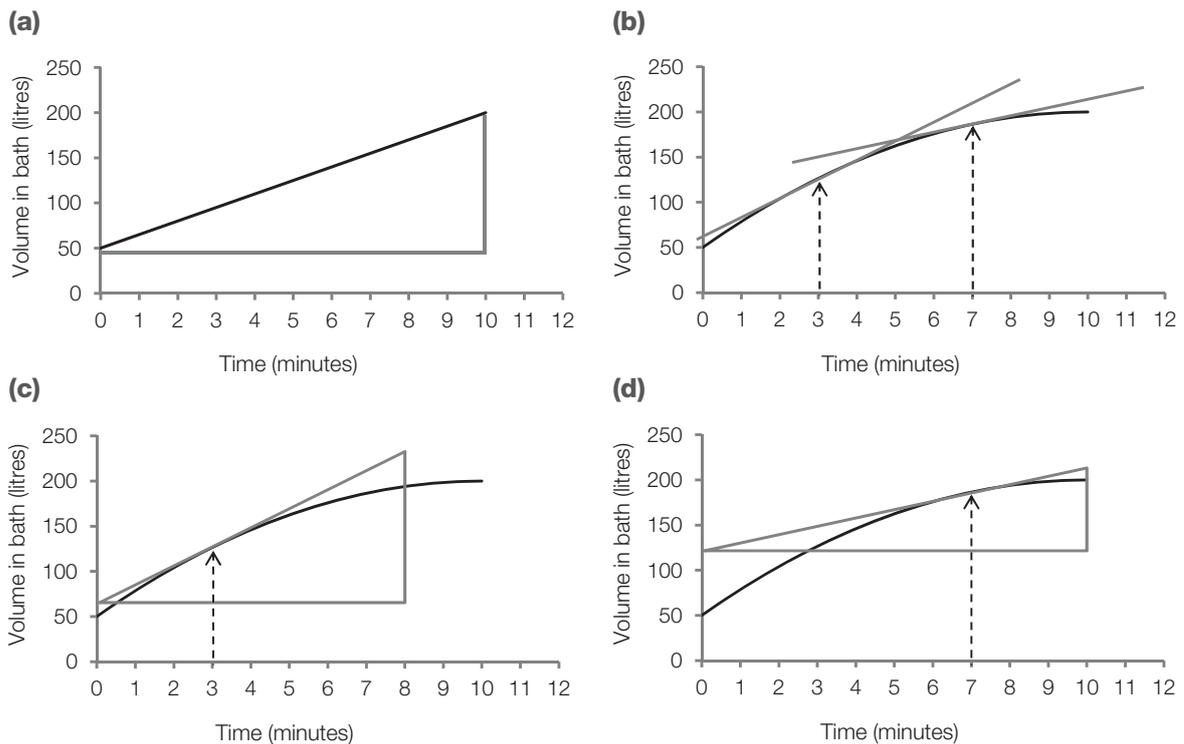
To draw a tangent by hand at a particular point on the curve, it is best to first mark this point on the curve. A ruler can then be positioned so that it passes through this point, with the curve on either side of this point sloping away from the ruler.

The gradients of these tangents can be calculated in the same way as before, by drawing a conveniently sized triangle, as shown in Figures 9.26c and 9.26d. The values of the gradient work out, in fact, at 21 litres/minute and 9 litres/minute respectively.

Drawing a tangent to a curve and calculating the gradient gives an *instantaneous* rate of change, i.e. the rate at that particular instant in time. It is *not* the same as the *average (mean)* rate up to that point in time. This would be found by dividing the total change in volume by the total time elapsed. It is important to make this distinction between *instantaneous* rate of change and *average* rate of change.

A similar distinction is important in the case of a graph of potential difference (*V*) against current (*I*) for a non-ohmic component, such as a filament bulb. Since it does not follow Ohm's Law, it is not a straight line graph but a *curve*. The resistance (*R*) of the component at any point is found by dividing the *value* of *V* by the *value* of *I* ($R = V/I$). However, it is sometimes believed, incorrectly, that it is the *gradient* at a point on the curve that gives the resistance (*R*) at that point. This is *not* the resistance but is the *instantaneous rate of change* of *V* with *I* (the *change* in *V* divided by the *change* in *I*). For a resistor that follows Ohm's Law, the graph is a straight line passing through the origin: calculating the gradient of the line gives the same value as calculating *V/I* for any pair of points along it.

9.13 Graphs of rates against time: area under the line

The previous section showed how it was possible to calculate a gradient at any point along a line. Suppose that this is done for a series of points along each of the lines in the graphs shown earlier in Figure 9.25. These gradients represent the **_rate_** of change of the volume of water in the bath at each of these points in time (i.e. the rate of flow of the water). If these values are then plotted against time, the **_line graphs_** obtained are shown in Figure 9.27.

The first of these graphs (Figure 9.27a) shows a horizontal straight line. This represents a *constant rate of flow* of water, with a value of 15 litres/minute.

The second graph (Figure 9.27b) also shows a straight line but here it slopes downwards. The flow rate starts at a high value (30 litres/minute) and then drops to zero after 10 minutes. Since it is a *straight line*, it means that the *rate of flow* decreases at a *constant rate* over this period of time. What we are talking about here is a *rate of change of a rate of change* – quite a complex idea! This idea is quite commonly encountered in 11–16 science, though in a different context: an *acceleration* is a rate of change of a rate of change of displacement (see Section 10.7 *Gradients of lines on speed–time and velocity–time graphs* on page 116).

Figure 9.27 Graphs of rates against time: a bath filling with water

(a)

(b)
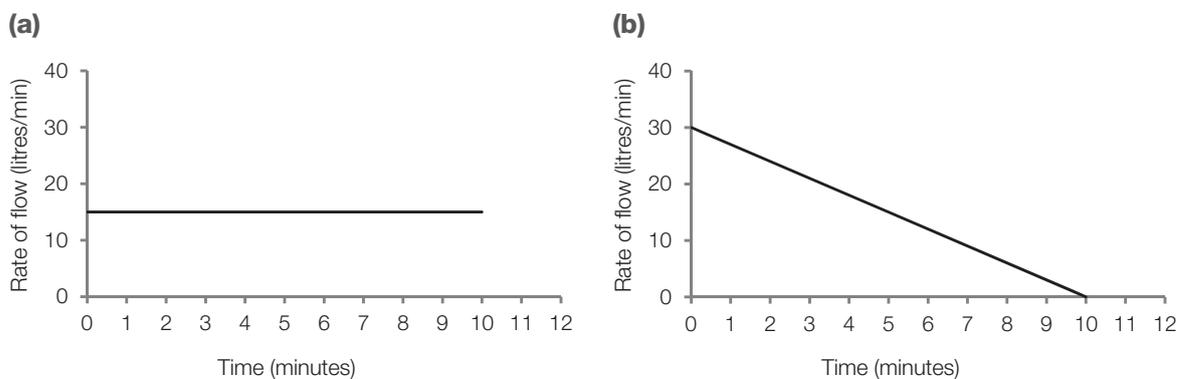
The previous section also showed how it is possible to use a graph showing a *quantity* plotted against *time* to calculate a *rate of change of the quantity*. It is also possible to go 'backwards'. In other words, it is possible to use a graph showing a *rate of change of a quantity* plotted against *time* to calculate the *quantity*.

Suppose we want to find the volume of water added to the bath between 3 minutes and 7 minutes. The graph in Figure 9.28a shows this period of time for the 'constant flow' bath. From this we can see that water flowed at a rate of 15 litres/minute for 4 minutes. To obtain the volume of water added in this time, we multiply these two values together, giving a total of 60 litres.

One way of thinking about this calculation is that it is the same as calculating the area of the shaded rectangle, i.e. the **_area under the line_** on the graph. In fact, for any line graph where the *line* represents the values for a *rate of change of a quantity*, the *area under the line* represents the value of the *quantity*.

Figure 9.28 Finding the value of a quantity by calculating the area under a line

(a)

(b)
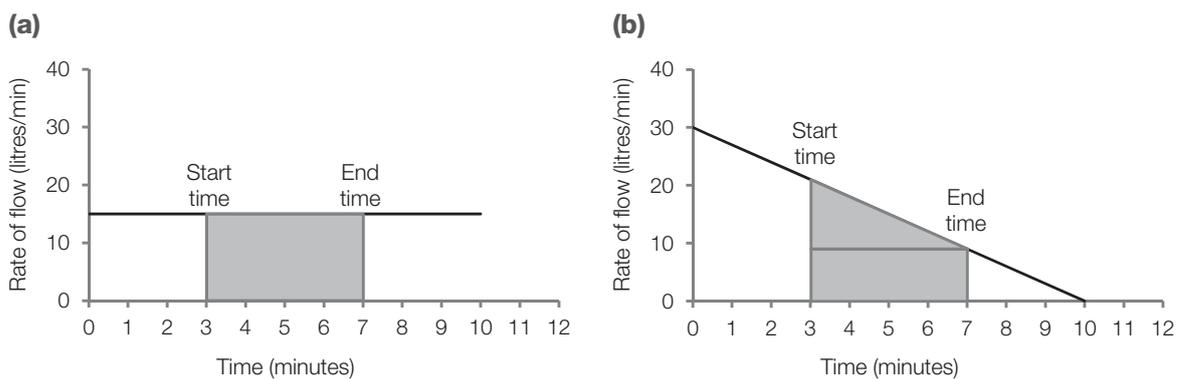
This idea, of calculating the area under the line on a graph also applies to the bath that is being filled with a *changing* rate of flow of water. This is shown in Figure 9.28b. Here, it is not quite so straightforward to calculate the area because it is not a rectangle.

One way of doing this is to split the area into two parts – a rectangle with a triangle on the top. The areas of these can then be found separately and added together. (The area of a right-angled triangle is one half of the area of a rectangle with the same base and height; see Section 10.2 *Length, area and volume* on page 108 for the formula to calculate the area of a right-angled triangle.)

Another way of doing this is to multiply the *mean* rate of flow by the time. On this graph, the mean rate of flow is the value at 5 minutes (it is the mean of the values at 3 and 7 minutes). This is equivalent to calculating the shape of the whole shaded area, which is a *trapezium*; pupils learn about calculating the area of a trapezium in mathematics.

Both of the graphs in Figure 9.28 are straight line graphs. If the graph had shown a *curve* then the area under the curve would still have the same meaning, though finding it would be less straightforward. One technique, if the graph is plotted on graph paper, is to estimate the numbers of large and small grid squares that are under the curve, and add up the areas.

Talking about baths filling with water is a concrete way of thinking about these ideas. In 11–16 science, however, pupils more often use calculations of the 'area under a line' in the context of *velocity–time graphs* (or speed–time graphs). These are discussed in more detail in Section 10.8 *Area under the line on speed–time and velocity–time graphs* on page 118.